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# The Lagrangian in classical field theory

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**Abstract.** A consideration of some simple examples of physical systems leads to a formulation of the most important results of Lagrangian theory in distribution-theoretical terms. From the action, defined as the value of the Lagrange distribution on a test function, come the equations of motion (in two different forms), the conserved energy tensor distribution, and the canonical momentum distribution.

## 1. Introduction

Traditionally (see e.g. Barut 1980), classical field theories are formulated by means of an action principle such as

$$\delta I = 0, \tag{1}$$

where

$$I = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3 x\, \mathscr{L}. \tag{2}$$

The Lagrangian density $\mathscr{L}$ (henceforth simply the 'Lagrangian') is a function of the fields $\psi$ in spacetime. The limits on the spatial integration are often left undefined, and at the same time one is a little vague about the allowable behaviour of the fields at infinity; the trouble is that the conditions required to make the integral exist exclude fields one would like to keep, like plane waves.

The rules for the variations $\delta\psi$ of the fields are imposed rather arbitrarily, but long usage has desensitised most people to their strangeness. Indeed, the whole scheme has become an almost entirely formal algorithm for obtaining equations of motion and, by Noether's theorem, the (stress–)energy(–momentum) tensor $T$. In formal work integrals do not need to exist and functions may be evaluated even where they are not defined. The most familiar instance of the latter is in the electrodynamic expression

$$j(x)\cdot A(x) = \int \mathrm{d}\tau\, v\cdot A(z)\delta(x-z)$$

for a charged point particle, where $A$ includes the self-field.

Many of these problems can be overcome by recasting the theory in a slightly different form. We continue to regard the fields $\psi$ as ordinary functions in spacetime. In certain circumstances we even accept some singularities in the $\psi$; the important thing is that they should be sufficient to specify the physical system. We think of the $\psi$ as the kinematical variables.

The Lagrangian $\mathscr{L}$ will be regarded as a distribution, that is, a linear functional on a space of test functions $\phi$. The functions $\phi$ in spacetime are infinitely differentiable and have compact support (vanish outside some finite region). Only the simplest properties of distributions, as described, for example, in the first few pages of Gel'fand and Shilov (1964), are used in the formulation. When $\mathscr{L}$ is determined by a smooth function (also called $\mathscr{L}$ when it exists) of $\psi$ and its derivatives, the distribution is regular, and its value on $\phi$ may be written

$$I(\phi)=(\mathscr{L},\phi)=\int \mathrm{d}^4x\,\phi\mathscr{L}. \tag{3}$$

The new form of the action is $I(\phi)$, a functional of both the kinematical variables $\psi$ and the test functions $\phi$.

Equation (3) is a simple generalisation of (2); indeed, the earlier form can be recovered, when it exists, from $(\mathscr{L},\phi)$ with a sequence of $\phi$'s. The behaviour of the fields $\psi$ at infinity is no longer a problem with (3) since the test functions $\phi$ vanish outside a finite region. By generalising from (3) to singular distributions $(\mathscr{L},\phi)$, that cannot be written as integrals over functions $\phi\mathscr{L}$, we have a simple method of extending the theory to cover such systems as point particles, and of avoiding problems such as undefined function values.

The formulation of the theory sketched here, in which the kinematical variables are ordinary functions, seems to be much simpler and perhaps to have more immediate contact with physical measurements than the formulation in Kurlandzki (1975), in which the kinematical fields $\psi$ are also distributions. Even in the present theory there is room for fields which are distributions (such as those that satisfy Maxwell's equations with a point particle source), but they appear at a different stage. An example of new distribution theory fields is provided below, where canonical momenta are derived from the Lagrangian. But, before such extras are introduced, the theory is founded on a conceptual base of ordinary functions, the position of a particle or the displacement of a string, etc.

The real objective in developing a dynamical theory from a Lagrangian is to produce, at the same time as the equations of motion, a conserved energy tensor. Working from (3) we can obtain the connection between the equations of motion and the conserved energy tensor by an elegant and transparent invariance argument, which need not be interpreted as a minimisation of the action (whose physical content remains obscure). We do not attempt to generalise the principle of least action, but use instead the invariance of the action when both the kinematical variables $\psi$ and the test functions $\phi$ are transformed.

In the following we consider several physical systems in the new formulation of Lagrangian theory: firstly, the massive scalar field, in which all densities are smooth and any mathematical operation that could conceivably be legitimate is allowed; secondly, the free point particle, which is singular only if one insists on considering it as a field. The energy tensors, giving the densities in spacetime, for both the point particle and the scalar field can be derived in exactly the same way. These simple examples help to define a suitable conceptual framework which is then available for more difficult problems when non-formal procedures are needed.

The theory based on (3) seems the appropriate setting in which to describe in a unified way the interaction of a point particle with a field. The point particle is bound to create singularities in the field which the formulation (1), (2) is ill equipped to deal with. The last example considered below is the vibrating string with an attached point mass; it illustrates very well how the distribution theory formulation unites point and field.

## 2. Massive scalar field

The Lagrangian for the real, massive scalar field $\psi(x)$ is

$$\mathscr{L} = -\tfrac{1}{2}[\partial\psi\cdot\partial\psi + m^2\psi^2] \tag{4}$$

(The gradient vector $\partial\psi$ has covariant components $\partial\psi/\partial x^\mu$; the metric tensor $\eta$ has diagonal components $-+++$.) The action on $\phi$ is simply the value of the regular distribution $\mathscr{L}$:

$$I(\phi) = -\int d^4x\, \phi(x)\tfrac{1}{2}[\partial\psi\cdot\partial\psi + m^2\psi^2]. \tag{5}$$

The action (5) is invariant under various transformations, among them the translations, in the direction of a fixed vector $b$, given by

$$\phi(x) \to \phi(x, \alpha) = \phi(x - \alpha b), \qquad \psi(x) \to \psi(x, \alpha) = \psi(x - \alpha b), \tag{6}$$

where $\alpha$ is any real number. In (6), the physical system $\psi$ is translated by $\alpha b$, as is the test function $\phi$. The latter transformation is a generalisation of the traditional changes of limits in (2).

Making the replacements (6) in (5) we get a function of $\alpha$ at the first stage, whose derivative is zero in view of the invariance of the action:

$$0 = dI/d\alpha = -\int d^4x\, \{(\partial\phi/\partial\alpha)\tfrac{1}{2}[\partial\psi\cdot\partial\psi + m^2\psi^2] + \phi[\partial\psi\cdot\partial\partial\psi/\partial\alpha + m^2\psi\partial\psi/\partial\alpha]\}.$$

Integrating by parts, and using

$$\partial\phi/\partial\alpha = -b\cdot\partial\phi, \qquad \partial\psi/\partial\alpha = -b\cdot\partial\psi, \tag{7}$$

we get (putting $\alpha = 0$ so $\phi$ and $\psi$ again mean what they did originally)

$$dI/d\alpha|_{\alpha=0} = 0 = \int d^4x\, \{b\cdot\partial\phi\tfrac{1}{2}[\partial\psi\cdot\partial\psi + m^2\psi^2] - \partial\phi\cdot\partial\psi\partial\psi\cdot b + \phi[\partial^2\psi - m^2\psi]\partial\psi/\partial\alpha\}. \tag{8}$$

The integrated terms vanish because $\phi$ has compact support.

If the equations of motion are satisfied,

$$\partial^2\psi - m^2\psi = 0,$$

then (8) becomes

$$0 = -\int d^4x\, b\cdot\{\partial\psi\partial\psi - \eta\tfrac{1}{2}[\partial\psi\cdot\partial\psi + m^2\psi^2]\}\cdot\partial\phi. \tag{9}$$

Equation (9) is the form taken by the distribution theory relation

$$0 = (\partial_\mu T^{\mu\nu}b_\nu, \phi) \equiv -(T^{\mu\nu}b_\nu, \partial_\mu\phi) \tag{10}$$

when the energy tensor $T$ is a regular distribution determined by the dyadic function

$$T=\partial\psi\partial\psi-\eta\tfrac{1}{2}[\partial\psi\cdot\partial\psi+m^2\psi^2]. \tag{11}$$

Equation (9) is the distribution theory form of the conservation of the energy tensor $T$, but since this distribution is regular and determined by the function in (11), we can verify the conservation independently from the derivative

$$\partial\cdot T=(\partial^2\psi)\partial\psi+(\partial\psi\cdot\partial\partial\psi)-[\partial\psi\cdot\partial\partial\psi+m^2\psi\partial\psi]=0,$$

where the equations of motion have been used.

## 3. Free point particle

By contrast with the case of the scalar field, the Lagrangian for the classical point particle is not a regular distribution, that is, not of the form (3). The traditional form for the action (Rohrlich 1965) is minus an integral of rest mass times proper time elapsed

$$I=-\int m\,\mathrm{d}\tau,$$

from which we generalise to the singular distribution with support only on the particle's worldline $z(\tau)$:

$$I(\phi)=-\int m\,\mathrm{d}\tau\,\phi(z(\tau)).$$

When we transform the integration variable to a dynamically neutral $\lambda$, the action takes the form

$$I(\phi)=-\int m\,\mathrm{d}\lambda\left(-\frac{\mathrm{d}z}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}z}{\mathrm{d}\lambda}\right)^{1/2}\phi(z(\lambda)). \tag{12}$$

Replacing in (12) $z(\lambda)$ by $z(\lambda,\alpha)$ and $\phi(x)$ by $\phi(x,\alpha)$, $I$ becomes a function of $\alpha$, with derivative

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\alpha}=&-\int m\,\mathrm{d}\lambda\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{1/2}\left\{\frac{\partial\phi}{\partial\alpha}+\frac{\partial z}{\partial\alpha}\cdot\partial\phi\right\}\\&+\int m\,\mathrm{d}\lambda\,\phi\frac{\partial^2 z}{\partial\alpha\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{-1/2}\\=&-\int m\,\mathrm{d}\lambda\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{1/2}\left\{\frac{\partial\phi}{\partial\alpha}+\frac{\partial z}{\partial\alpha}\cdot\partial\phi\right\}\\&-\int m\,\mathrm{d}\lambda\,\phi\frac{\partial z}{\partial\alpha}\cdot\frac{\partial}{\partial\lambda}\left[\frac{\partial z}{\partial\lambda}\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{-1/2}\right]\\&-\int m\,\mathrm{d}\lambda\frac{\partial z}{\partial\lambda}\cdot\partial\phi\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\alpha}\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{-1/2}.\end{aligned}$$

If $\phi(x,\alpha)$ and $z(\lambda,\alpha)$ represent translations $\alpha b$, so that

$$\phi(x,\alpha)=\phi(x-\alpha b),\qquad z(\lambda,\alpha)=z(\lambda)+\alpha b,$$

then the action is invariant, $dI/d\alpha = 0$, and $\partial\phi/\partial\alpha + \partial\phi \cdot \partial z/\partial\alpha = 0$. If the equations of motion are satisfied too,

$$\frac{\partial}{\partial\lambda}\left[\frac{\partial z}{\partial\lambda}\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{-1/2}\right] = 0,$$

we get, for an arbitrary vector $b$,

$$0 = -\int m \, d\lambda \, b \cdot \frac{\partial z}{\partial\lambda}\frac{\partial z}{\partial\lambda} \cdot \partial\phi\left(-\frac{\partial z}{\partial\lambda}\cdot\frac{\partial z}{\partial\lambda}\right)^{-1/2}. \tag{13}$$

Equation (13) is just the distribution theory expression of the conservation of the energy tensor given by the singular distribution

$$(T, \phi) = \int m \, d\lambda \frac{dz}{d\lambda}\frac{dz}{d\lambda}\left(-\frac{dz}{d\lambda}\cdot\frac{dz}{d\lambda}\right)^{-1/2}\phi(z) = \int m \, d\tau \, vv\phi(z). \tag{14}$$

The informal representation of $T$ is the more familiar

$$T = \int d\tau \, mvv\delta(x - z(\tau)). \tag{15}$$

Further discussion of the energy tensor for a point particle is given, for example, in Rowe (1983).

## 4. Vibrating string

We now consider a transversely vibrating string of uniform density $\rho$ and constant tension $F$, at first without the attached point mass that gives the system its real interest. The traditional theory is fully developed in Corben and Stehle (1960). The Lagrangian is

$$\mathscr{L} = \tfrac{1}{2}(\rho[\partial\psi/\partial t]^2 - F[\partial\psi/\partial x]^2) \equiv \tfrac{1}{2}(\rho\psi_t^2 - F\psi_x^2). \tag{16}$$

When the usual wave equation is satisfied, the invariance of the action

$$I(\phi) = \int dt \int dx \, \mathscr{L}\phi \tag{17}$$

leads quickly to the energy and longitudinal momentum conservation laws in a similar way to that used with (5):

$$(\partial/\partial t)(\tfrac{1}{2}\rho\psi_t^2 + \tfrac{1}{2}F\psi_x^2) + (\partial/\partial x)(-F\psi_t\psi_x) = 0, \tag{18}$$

$$(\partial/\partial t)(-\rho\psi_t\psi_x) + (\partial/\partial x)(\tfrac{1}{2}\rho\psi_t^2 + \tfrac{1}{2}F\psi_x^2) = 0. \tag{19}$$

If we now attach a point mass $m$ to the string at a fixed coordinate $x = X$ (so it can move transversely only), the new action is no longer the value of a regular distribution as in (17), but is composed of two pieces, of which one has only point support in $x$:

$$I(\phi) = \int dt\left(\tfrac{1}{2}m\psi_t^2(X, t)\phi(X, t) + \int dx \, \phi\tfrac{1}{2}[\rho\psi_t^2 - F\psi_x^2]\right). \tag{20}$$

A second important change is that the displacement $\psi(x, t)$ is no longer smooth—$\psi$

is continuous but $\partial\psi/\partial x$ has a jump at $x=X$. So the $x$ integration in (20) must be regarded as being over two separate ranges $x<X$ and $x>X$ when we integrate by parts.

The action (20) is invariant under the time translation

$$\phi(x,t)\to\phi(x,t-\alpha),\qquad \psi(x,t)\to\psi(x,t-\alpha),\qquad X\to X.$$

Making the replacement, differentiating, and integrating by parts gives

$$\int \mathrm{d}t\Big[\phi(X,t)\psi_t(X,t)[m\psi_{tt}(X,t)-F\psi_x(x,t)|_{X-}^{X+}]+\tfrac{1}{2}m\psi_t^2(X,t)\frac{\partial\phi}{\partial t}(X,t)$$
$$+\int \mathrm{d}x\Big(\phi\psi_t(\rho\psi_{tt}-F\psi_{xx})+\frac{\partial\phi}{\partial t}\frac{1}{2}[\rho\psi_t^2+F\psi_x^2]+\frac{\partial\phi}{\partial x}(-F\psi_x\psi_t)\Big)\Big]=0. \quad (21)$$

In order to extract conservation of energy from (21), two equations of motion must be satisfied, one for the continuum motion of the string (coefficient of $\phi(x,t)$) and one for the motion of the point at $x=X$ (coefficient of $\phi(X,t)$):

$$\rho\psi_{tt}-F\psi_{xx}=0,\qquad (x\neq X), \quad (22)$$

$$m\psi_{tt}=F\psi_x|_{X-}^{X+},\qquad (x=X). \quad (23)$$

It is an attractive feature of the formulation that these two types of equation appear in a unified way.

From the remainder of (21) we obtain energy conservation; the energy density distribution is given by

$$(T^{00},\phi)=\int \mathrm{d}t\Big(\tfrac{1}{2}m\psi_t^2(X,t)\phi(X,t)+\int \mathrm{d}x\,\phi\tfrac{1}{2}[\rho\psi_t^2+F\psi_x^2]\Big) \quad (24)$$

and the energy flux distribution by

$$(T^{10},\phi)=\iint \mathrm{d}t\,\mathrm{d}x\,\phi(-F\psi_x\psi_t). \quad (25)$$

Since $\psi_x$ is discontinuous, it is important that the conservation law be expressed in distribution theory form

$$-(T^{00},\partial\phi/\partial t)-(T^{10},\partial\phi/\partial x)=0. \quad (26)$$

The second translation invariance of the action (20) is for

$$\phi(x,t)\to\phi(x-\alpha,t),\qquad \psi(x,t)\to\psi(x-\alpha,t),\qquad X\to X+\alpha.$$

Working as before we find

$$\mathrm{d}I/\mathrm{d}\alpha|_{\alpha=0}=0=\int \mathrm{d}t\,\phi(X,t)[\tfrac{1}{2}\rho\psi_t^2+\tfrac{1}{2}F\psi_x^2|_{X-}^{X+}$$
$$+\iint \mathrm{d}t\,\mathrm{d}x\{-(\partial\phi/\partial x)\tfrac{1}{2}(\rho\psi_t^2+F\psi_x^2)$$
$$+(\partial\phi/\partial t)(\rho\psi_t\psi_x)+\phi\psi_x(\rho\psi_{tt}-F\psi_{xx})\}. \quad (27)$$

Both the wave equation and a subsidiary condition

$$[\tfrac{1}{2}\rho\psi_t^2+\tfrac{1}{2}F\psi_x^2|_{X-}^{X+}=0 \quad (28)$$

must be satisfied in order that momentum be conserved in the form

$$-\iint [(\partial\phi/\partial t)(-\rho\psi_t\psi_x)+(\partial\phi/\partial x)\tfrac{1}{2}(\rho\psi_t^2+F\psi_x^2)]\,\mathrm{d}x\,\mathrm{d}t=0. \tag{29}$$

It is again important, because of the singularity in $\psi$, that the conservation law be expressed in distribution theory form. The subsidiary condition ensures that the particle receives no longitudinal force, a necessary restriction since we have taken $X$ to be fixed.

A second way of expressing the equations of motion (22) and (23) is by means of a distribution theory differential equation for the canonical momentum.

The components $p^0$, $p^1$ of the canonical momentum may be defined as follows. We replace in the action (20) the derivatives of the kinematical variables $\psi_t$ (or $\psi_x$) by smooth variations

$$\psi_t(x,t)\to\psi_t(x,t)+\alpha\phi_0(x,t)$$

(or, respectively, $\psi_x\to\psi_x+\alpha\phi_0$). The function $\phi_0$ should be infinitely differentiable so that the product $\phi_0\phi$ is also a test function if $\phi$ is. The definition of $p^0$ is then

$$(p^0,\phi_0\phi)=\frac{\partial}{\partial\alpha}(\mathscr{L},\phi)\bigg|_{\alpha=0}=\int \mathrm{d}t\left(m\psi_t(X,t)\phi_0\phi(X,t)+\int \mathrm{d}x\,\rho\psi_t\phi_0\phi\right) \tag{30}$$

and, similarly,

$$(p^1,\phi_0\phi)=\partial(\mathscr{L},\phi)/\partial\alpha|_{\alpha=0}=\int \mathrm{d}t\,\mathrm{d}x(-F\psi_x\phi_0\phi). \tag{31}$$

It is easy to check that the equation

$$(\partial p^0/\partial t+\partial p^1/\partial x,\phi)\equiv-(p^0,\partial\phi/\partial t)-(p^1,\partial\phi/\partial x)=0 \tag{32}$$

is satisfied, and it is equivalent to both (22) and (23).

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